

DISCRETE SERIES OF SIMPLE LIE GROUPS WITH ADMISSIBLE RESTRICTION TO A $SL_2(\mathbb{R})$

ESTHER GALINA, JORGE A. VARGAS

ABSTRACT. In this note we determine the irreducible square integrable representations of a simple group which admits an admissible restriction to a subgroup H locally isomorphic to $SL_2(\mathbb{R})$. We show such representation is holomorphic and we determine the essentially unique H with this property as well as multiplicity formulae.

1. INTRODUCTION

Let G be a connected simple matrix Lie group and K a fixed maximal compact subgroup of G . We assume that both groups K and G has the same rank. From now on $T \subset K$ is a fixed maximal torus. Therefore, T is a compact Cartan subgroup of G . Under these hypothesis, Harish-Chandra showed there exists irreducible unitary representations of G so that its matrix coefficients are square integrable with respect to a Haar measure on G . Let H denote the image of a nontrivial continuous morphism from $SL_2(\mathbb{R})$ into G . The aim of this note is to determine the pairs $((\pi, V), H)$ of irreducible square integrable representations (π, V) of G which admits admissible restriction to H . That is, $\text{res}_H \pi$ is an admissible representation, or equivalently, $\text{res}_H \pi$ is equal to a discrete Hilbert sum of irreducible representations of H and the multiplicity of each irreducible factor is finite.

To state the main results we need to fix some more notation. The Lie algebra of a Lie group is denoted by the corresponding lower case Fraktur font and the complexification of a real Lie algebra, or a vector space, is denoted by adding the subscript \mathbb{C} . The conjugation of $X \in \mathfrak{g}_{\mathbb{C}}$ with respect to the real form \mathfrak{g} of $\mathfrak{g}_{\mathbb{C}}$ is denoted by \bar{X} . Denote by $\Phi(\mathfrak{g}, \mathfrak{t})$ (resp. $\Phi(\mathfrak{k}, \mathfrak{t})$) the root system of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathfrak{k}_{\mathbb{C}}$ in $\mathfrak{k}_{\mathbb{C}}$). The set of noncompact roots is defined by $\Phi_n = \Phi(\mathfrak{g}, \mathfrak{t}) \setminus \Phi(\mathfrak{k}, \mathfrak{t})$. We write $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ the Cartan decomposition corresponding to the pair (\mathfrak{g}, K) . Then, the root vectors associated to $\alpha \in \Phi_n$ (resp. $\alpha \in \Phi(\mathfrak{k}, \mathfrak{t})$) lie in $\mathfrak{p}_{\mathbb{C}}$ (resp. in $\mathfrak{k}_{\mathbb{C}}$). Recall that a system of positive roots Ψ of the root system $\Phi(\mathfrak{g}, \mathfrak{t})$ is *holomorphic* whenever the sum $\alpha + \beta$ is not a root for every pair $\alpha, \beta \in \Psi_n := \Psi \cap \Phi_n$. For a unitary representation (π, V) , the action of $X \in \mathfrak{g}_{\mathbb{C}}$ on the subspace

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of K -finite vectors is denoted by $\dot{\pi}(X)$. A unitary irreducible representation (π, V) is *holomorphic* when there exists a holomorphic system of positive roots $\Psi \subset \Phi(\mathfrak{g}, \mathfrak{t})$ and a nonzero K -finite vector $v \in V$ so that $\dot{\pi}(X_{-\gamma})v = 0$ for every noncompact root γ in Ψ and for every root element X_γ associated to γ .

After the work of Harish-Chandra and other researchers it has been shown that holomorphic system of positive roots, as well as holomorphic unitary irreducible representations, do exist if and only if the Riemannian manifold G/K admits a G -invariant complex structure. That is, G/K is a Hermitian symmetric space.

We are ready to state our first result.

Theorem 1.1. *Assume there exists a square integrable irreducible representation π for G and a subgroup H locally isomorphic to $SL_2(\mathbb{R})$ such that the restriction of π to H is an admissible representation. Then, G/K is an Hermitian symmetric space and π is a holomorphic representation of G .*

In order to list the pairs $((\pi, V), H)$ we are searching for, we recall on one side the list of pairs of Hermitian symmetric Lie algebras $(\mathfrak{g}, \mathfrak{k})$ and a classification, up to conjugacy, of the subgroups of G locally isomorphic to $SL_2(\mathbb{R})$. The list of Hermitian symmetric pairs is:

$AIII$ ($\mathfrak{su}(p, q), \mathfrak{su}(p) \oplus \mathfrak{su}(q) + \mathfrak{so}(2)$)
 BDI ($\mathfrak{so}(p, 2), \mathfrak{so}(p) \oplus \mathfrak{so}(2)$)
 CI ($\mathfrak{sp}(n, \mathbb{R}), \mathfrak{u}(n)$)
 $DIII$ ($\mathfrak{so}^*(2p), \mathfrak{u}(p)$)
 $EIII$ ($\mathfrak{e}_{6(-14)}, \mathfrak{so}(5) \oplus \mathfrak{so}(2)$)
 $EVII$ ($\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 \oplus \mathfrak{so}(2)$)

Among them, only $AIII(p = q)$, BDI , CI , $DIII$ (p even) and $EVIII$ corresponds to tube domains.

The classification of the Lie subalgebras \mathfrak{h} of \mathfrak{g} isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, up to conjugacy, has been obtained by Kostant-Rallis. In particular, they show that the number of conjugacy classes is finite. An explicit list of representatives of the conjugacy classes for group G has been obtained by [Oh], [Dk2] for classical real Lie groups and in [Dk1] for the exceptional groups. In order to state the version of the classification more suitable for this work, to the end of this introduction we assume

G/K is a Hermitian symmetric space.

We fix a holomorphic system of positive roots $\Psi \subset \Phi(\mathfrak{g}, \mathfrak{t})$, and we denote the simple roots of Ψ by

$$(1.1) \quad \{\beta_1, \dots, \beta_\ell\} \quad \text{where } \beta_1, \dots, \beta_{\ell-1} \in \Phi(\mathfrak{k}, \mathfrak{t}) \text{ and } \beta_\ell \in \Phi_n.$$

The highest root is written as $\beta_M = \sum_j c_j \beta_j$ with $c_j \geq 1$ for all j and $c_\ell = 1$.

Under this setting, they have shown that a subalgebra of \mathfrak{g} isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ is conjugated by an element of G to a subalgebra \mathfrak{h} whose complexification is spanned by a normal triple, or more precisely, a KS-triple

$$Z_{\mathfrak{h}} \in i(\mathfrak{t} \cap \mathfrak{h}), \quad E_{\mathfrak{h}} \in \mathfrak{p}_{\mathbb{C}} \cap \mathfrak{h}_{\mathbb{C}}, \quad F_{\mathfrak{h}} \in \mathfrak{p}_{\mathbb{C}} \cap \mathfrak{h}_{\mathbb{C}}$$

which satisfies,

$$(1.2) \quad \begin{aligned} [Z_{\mathfrak{h}}, E_{\mathfrak{h}}] &= 2E_{\mathfrak{h}}, \quad [Z_{\mathfrak{h}}, F_{\mathfrak{h}}] = -2F_{\mathfrak{h}}, \quad [E_{\mathfrak{h}}, F_{\mathfrak{h}}] = Z_{\mathfrak{h}}, \quad \bar{E}_{\mathfrak{h}} = F_{\mathfrak{h}} \\ \alpha(Z_{\mathfrak{h}}) &\geq 0 \quad \text{for every } \alpha \in \Psi_c := \Psi \cap \Phi(\mathfrak{k}, \mathfrak{t}) \end{aligned}$$

Moreover, the characteristic vector $Z_{\mathfrak{h}}$ of the *KS*-triple determines the $K_{\mathbb{C}}$ -conjugacy class of \mathfrak{h} . That is, if $Z_{\mathfrak{h}} = Z_{\mathfrak{h}'}$ then $\mathfrak{h}_{\mathbb{C}}$ is $K_{\mathbb{C}}$ -conjugated to \mathfrak{h}' (see Theorem 9.4.4 of [CM]).

The finite set of vectors $Z_{\mathfrak{h}}$ which parametrize the G -conjugacy classes of SL_2 -subgroups, can be expressed explicitly in terms of the dual basis to the basis of simple roots (1.1). The coordinates of each $Z_{\mathfrak{h}}$ has been computed for the classical Lie algebras in [Dk2], [Oh] and for the exceptional ones in [Dk1]. Moreover, for the classical Lie algebras, they compute the eigenvalues of the matrix $Z_{\mathfrak{h}}$. More precisely, if $Z_j, j = 1, \dots, \ell$, denote the dual basis to (1.1). That is, the elements of $\mathfrak{t}_{\mathbb{C}}$, Z_j are such that $\beta_r(Z_s) = \delta_{rs}$, hence, we have

$$Z_{\mathfrak{h}} = \sum_{1 \leq s \leq \ell} \beta_s(Z_{\mathfrak{h}}) Z_s$$

It follows from (1.2) and the work of Kostant-Rallis that $\beta_s(Z_{\mathfrak{h}})$ is a non negative integer for $s = 1, \dots, \ell-1$ and that $\beta_{\ell}(Z_{\mathfrak{h}})$ is an integer. The precise values of the integers $\beta_s(Z_{\mathfrak{h}})$ for the exceptional Lie algebras are listed in [Dk1]. In Proposition 3.5 we show that there is a unique characteristic vector $Z_{\mathfrak{h}}$ with $\beta_{\ell}(Z_{\mathfrak{h}})$ a positive number. We also show that the hypothesis that (π, V) is a holomorphic square integrable representation with admissible restriction to H forces either that $\beta_{\ell}(Z_{\mathfrak{h}}) > 0$, or $\beta_{\ell}(Z_{\mathfrak{h}}) < 0$ and $\beta_i(Z_{\mathfrak{h}}) = 0$ for all $1 \leq i \leq \ell-1$. Next, we produce an explicit example of a KS-triple $\{Z_0, E_0, F_0\}$ such that any holomorphic representation π restricted to the corresponding SL_2 -subgroup H_0 is admissible. Later on we show that H_0 is unique up to conjugacy by elements of G .

We denote by (\cdot, \cdot) the inner product on $i\mathfrak{t}^*$ associated to the Killing form on \mathfrak{g} . From the holomorphic system Ψ , Harish-Chandra has constructed a strongly orthogonal spanning set

$$(1.3) \quad S = \{\gamma_1, \dots, \gamma_r\} \subset \Psi_n$$

as follows: $\gamma_1 = \beta_M$ is the maximal root in $\Psi_n = \Psi \cap \Phi_n$, γ_2 is the maximal root in $\{\gamma \in \Psi_n : \gamma \perp \gamma_1\}$ and so on. Here r is the real rank of the group G . Notice that there is no non compact root orthogonal to S . For each root $\gamma \in \Psi_n$ we fix a KS-triple $Z_{\gamma} \in i\mathfrak{t}$, $E_{\gamma} \in \mathfrak{g}_{\gamma} \cap \mathfrak{p}_{\mathbb{C}}$ and $F_{\gamma} \in \mathfrak{g}_{\gamma} \cap \mathfrak{p}_{\mathbb{C}}$. In particular, if H_{γ} denotes the coroot vector of γ , then $Z_{\gamma} = \frac{2H_{\gamma}}{(\gamma, \gamma)}$, $\bar{E}_{\gamma} = F_{\gamma}$ and $[E_{\gamma}, F_{\gamma}] = Z_{\gamma}$. Let

$$(1.4) \quad Z_0 = Z_{\gamma_1} + \dots + Z_{\gamma_r}, \quad E_0 = E_{\gamma_1} + \dots + E_{\gamma_r}, \quad F_0 = F_{\gamma_1} + \dots + F_{\gamma_r}.$$

Since S is a strongly orthogonal set of non compact roots it readily follows that $\{Z_0, E_0, F_0\}$ is a KS-triple. Let H_0 the image of $SL_2(\mathbb{R})$ in G associated to this triple. Later on, in Lemma 3.3, we show that $\beta_\ell(Z_0) > 0$. This will verify the first statement in the following theorem.

Theorem 1.2.

- (i) *Any holomorphic irreducible, square integrable representation of G has an admissible restriction to H_0 .*
- (ii) *Whenever a holomorphic irreducible, square integrable representation of G has admissible restriction to a subgroup H locally isomorphic to $SL_2(\mathbb{R})$, then H is conjugated under G to H_0 .*

The first statement follows from work of Vergne, Jacobsen, T. Kobayashi, Möllers and Oshima, for a reference [OM]. However, we give an independent proof. In section 5 we compute the multiplicity and Harish-Chandra parameter of each irreducible H_0 -factor.

This note is organized as follows, in the introduction we state our main results but the one on multiplicity and Harish-Chandra parameter of each irreducible factor, which is presented in Section 5. In section 2 and 3 we respectively present the proof of the two theorems stated in the introduction. In section 4 we write down the groups involved in the statement of the results.

2. PROOF OF THEOREM 1.1

For each noncompact root β , the subalgebra $\mathfrak{sl}_2(\beta)$ of \mathfrak{g} spanned by the root elements associated to $\pm\beta$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ and invariant under the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to \mathfrak{g} . Thus, the real points of the subalgebra $\mathfrak{sl}_2(\beta)$ is an algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Let H_β denote the analytic subgroup corresponding to this real subalgebra.

Let H be as in Theorem 1.1 and T a compact Cartan subgroup of G as in the introduction. We may assume, after conjugation by an element of G , that $H \cap T = H \cap K$ is a maximal compact subgroup of H . The hypothesis that (π, V) restricted to H is admissible and π is square integrable, imply that π restricted to $H \cap T$ is an admissible representation, see [DV]. Since H_β is invariant under conjugation by elements of T , TH_β is an analytic subgroup of G with T as a maximal compact subgroup. To continue we formulate a useful result of Kobayashi.

Fact 2.1. For a given a closed subgroup L of G with finitely many connected component and invariant under the Cartan involution, we know that a maximal compact subgroup of L is $L \cap K$. In this context, if (π, V) is a irreducible representation of G and the restriction of π to $L \cap K$ is admissible, Theorem 1.2 of [Kb1], guarantees that the restriction of π to L is admissible.

Owing to the Fact 2.1 we have that π restricted to TH_β is an admissible representation. Moreover, Kobayashi shows that the subspace of T -finite vectors in V is identical to the subspace of K -finite vectors (see [Kb2], Proposition 1.6). The hypothesis (π, V) is a square integrable representation, implies that each of the TH_β - irreducible constituents is a square integrable representation of TH_β , finally, the Harish-Chandra module underlying an irreducible square integrable representations of $SL_2(\mathbb{R})$ is a Verma modules. Therefore, there exists a non zero K -finite vector $v \in V$ so that either $\dot{\pi}(X_{-\beta})(v) = 0$ or $\dot{\pi}(X_\beta)(v) = 0$. From Lemma 3.4 of [Vo] we have that $\dot{\pi}(X_{\pm\gamma})$, with $\gamma \in \Psi_n$, is injective for every nilpotent element unless G/K is a Hermitian symmetric space. In turn, it follows that (π, V) is either a holomorphic representation or antiholomorphic representation, details can be found in [Va]. Thus, the proof of Theorem 1.1 is finished.

3. PROOF OF THEOREM 1.2

Let \mathfrak{p}^+ denote the subspace spanned by the root elements corresponding to the roots in Ψ_n . Then, \mathfrak{p}^+ is invariant under the action of $K_{\mathbb{C}}$ via the adjoint representation of $G_{\mathbb{C}}$. Hence, the symmetric algebra $S(\mathfrak{p}^+)$ is a K -module. Let W denote the lowest K -type of the holomorphic discrete series representation (π, V) , then the space of K -finite vectors of π is K -isomorphic to the K -module $S(\mathfrak{p}^+) \otimes W$.

Fact 3.1. In [Kb1], [DV], it is pointed out that the restriction of π to L , with L a closed connected subgroup of K , is admissible if and only if the algebra of invariants $S(\mathfrak{p}^+)^L$ consists of the set of constant elements.

We apply this criterium to understand admissible restriction to the subgroup $H \cap T = \exp(\mathbb{R}iZ_{\mathfrak{h}})$ where H is locally isomorphic to $SL_2(\mathbb{R})$ as in Theorem 1.1 and (1.2).

Proposition 3.2. *A holomorphic discrete series representation (π, V) of G restricted to the subgroup $H \cap T = \exp(\mathbb{R}iZ_{\mathfrak{h}})$ is admissible if and only if one of the two following condition holds,*

- (i) $\beta_\ell(Z_{\mathfrak{h}})$ is positive, or
- (ii) $\beta_\ell(Z_{\mathfrak{h}})$ is negative and $\beta_j(Z_{\mathfrak{h}}) = 0$ for all $1 \leq j \leq \ell - 1$.

Proof. Indeed, the weights of $t_{\mathbb{C}}$ in $S(\mathfrak{p}^+)$ are $\sum_{\gamma \in \Psi_n} c_\gamma \gamma$ with c_γ a non-negative integer for every γ . Any root $\gamma \in \Psi_n$ is equal to β_ℓ plus a linear combination with non negative coefficients of the roots $\beta_1, \dots, \beta_{\ell-1}$. Thus, owing to (1.1), we have the inequalities

$$\beta_\ell(Z_{\mathfrak{h}}) \leq \gamma(Z_{\mathfrak{h}}) \leq \beta_M(Z_{\mathfrak{h}}), \text{ for } \gamma \in \Psi_n.$$

Hence,

$$\left(\sum_{\gamma \in \Psi_n} c_\gamma \right) \beta_\ell(Z_{\mathfrak{h}}) \leq \sum_{\gamma \in \Psi_n} c_\gamma \gamma(Z_{\mathfrak{h}}) \leq \left(\sum_{\gamma \in \Psi_n} c_\gamma \right) \beta_M(Z_{\mathfrak{h}})$$

Thus, according to the Fact 3.1, the restriction of π to $\exp(\mathbb{R}iZ_{\mathfrak{h}})$ is admissible if and only if

$$\sum_{\gamma \in \Psi_n} c_{\gamma} \gamma(Z_{\mathfrak{h}}) \neq 0 \quad \text{for every} \quad \sum_{\gamma \in \Psi_n} c_{\gamma} > 0$$

Hence, the converse implication in the proposition is true.

For the direct implication, if $\beta_{\ell}(Z_{\mathfrak{h}}) = 0$ then the vectors $E_{\beta_{\ell}}^k \in S(\mathfrak{p}^+)$, $k = 1, 2, \dots$, span an infinite dimensional subspace of $\mathfrak{t} \cap \mathfrak{h}$ -weight zero, which contradicts the admissible restriction of π to $T \cap H$. Thus, $\beta_{\ell}(Z_{\mathfrak{h}}) \neq 0$. If $\beta_{\ell}(Z_{\mathfrak{h}}) < 0$ there are two possibilities: either $\beta_k(Z_{\mathfrak{h}}) > 0$ for some $1 \leq k \leq \ell - 1$, or $\beta_k(Z_{\mathfrak{h}}) = 0$ for all $1 \leq k \leq \ell - 1$. In the first situation we recall that $\beta_j(Z_{\mathfrak{h}})$ is a nonnegative integer for any $j \leq \ell - 1$ and hence the space of vectors of T -weights $\nu = n(-\beta_{\ell}(Z_{\mathfrak{h}})\beta_k + \beta_k(Z_{\mathfrak{h}})\beta_{\ell})$, $n = 0, 1, 2, \dots$, is infinite dimensional and consists of vectors of $\mathfrak{t} \cap \mathfrak{h}$ -weight zero. This contradicts that π is an admissible representation of $\mathfrak{t} \cap \mathfrak{h}$ and implies that the second situation holds. \square

We observe that whenever the situation (ii) holds, there is another KS-triple which satisfies (1.2) for \mathfrak{h} , namely, the triple $\{-Z_{\mathfrak{h}}, F_{\mathfrak{h}}, E_{\mathfrak{h}}\}$. Now, $\beta_{\ell}(-Z_{\mathfrak{h}}) > 0$ and $\beta_j(-Z_{\mathfrak{h}}) = 0$, $j = 1, \dots, \ell - 1$, hence we are in situation (i). Note that that both KS-triples generate the real Lie algebra \mathfrak{h} .

For the particular element Z_0 defined in (1.4), we obtain the following result in relation with Proposition 3.2.

Lemma 3.3. *Let Z_0 be as in (1.4), then for every noncompact root β in Ψ , $\beta(Z_0) > 0$. In particular, for the simple noncompact root, $\beta_{\ell}(Z_0) > 0$.*

Proof. Since the system Ψ is holomorphic, $\beta + \gamma_j$ never is a root. Hence, for every j , we have $\frac{2(\beta, \gamma_j)}{(\gamma_j, \gamma_j)} \geq 0$. By construction, S is a strongly orthogonal spanning set. Hence, β is not orthogonal to some root in S , which shows the claim. \square

Considering Lema 3.3, Proposition 3.2 and Fact 2.1, we obtain the first statement of Theorem 1.2 as a corollary.

Corollary 3.4. *Let (π, V) be a holomorphic discrete series and H_0 the real subgroup of G associated to the KS-triple as in (1.4), then π restricted to $H_0 \cap T$ is admissible. Therefore, the restriction of π to H_0 is admissible.*

Let \mathfrak{h} a copy of $\mathfrak{sl}_2(\mathbb{R})$ as subalgebra of \mathfrak{g} whose complexification is generated by the triple (1.2). Henceforth, we assume $Z_{\mathfrak{h}}$ satisfies the hypothesis of Proposition 3.2. Then, any holomorphic irreducible square integrable representation (π, V) restricted to $\exp(i\mathbb{R}Z_{\mathfrak{h}}) = H \cap T$ is an admissible representation. Owing to Fact 3.1, π is an admissible representation of H . To continue to the proof of Theorem 1.2, we show the following.

Proposition 3.5. *Let $Z_{\mathfrak{h}}$ be a characteristic vector as in (1.2) and assume that $\beta_{\ell}(Z_{\mathfrak{h}})$ is positive. Then, $Z_{\mathfrak{h}} = Z_0$ and H is conjugated to H_0 by an element of G .*

For groups of type $EIII, EVII$ the equality $Z_{\mathfrak{h}} = Z_0$ follows by inspection of tables $X, XIII$ in [Dk1]. We proceed in a proof for all groups G , for this we need some other results to start with.

We recall a result of Barbasch-Vogan (Lemma 3.7.3 in [CM]) on the centralizer $\mathfrak{g}_{\mathbb{C}}^{E_{\mathfrak{h}}}$ of $E_{\mathfrak{h}}$ in $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}}$ denote the centralizer of \mathfrak{h} in $\mathfrak{g}_{\mathbb{C}}$. Thus, $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}}$ is a reductive subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let \mathfrak{u} denote the nilpotent radical of $\mathfrak{g}_{\mathbb{C}}^{E_{\mathfrak{h}}}$, then the result of Barbasch-Vogan establishes the direct sum decomposition

$$\mathfrak{g}_{\mathbb{C}}^{E_{\mathfrak{h}}} = \mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} + \mathfrak{u}$$

Since \mathfrak{h} is invariant under the Cartan involution associated to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we have that $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}}$ is also invariant under the Cartan involution. Moreover, $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}}$ is invariant under the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to the real algebra \mathfrak{g} . Therefore, we have the decompositions,

$$\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} = \mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} \cap \mathfrak{k}_{\mathbb{C}} + \mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} \cap \mathfrak{p}_{\mathbb{C}} \quad \mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} = (\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} \cap \mathfrak{g}) + i(\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} \cap \mathfrak{g})$$

and

$$\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} \cap \mathfrak{g} = \mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} \cap \mathfrak{k} + \mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} \cap \mathfrak{p}$$

In particular, we can conclude the following.

Lemma 3.6. *Let \mathfrak{h} satisfying the conditions of Proposition 3.2, then $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}}$ is a subalgebra of $\mathfrak{k}_{\mathbb{C}}$.*

Proof. Obviously, we have

$$\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} \subset \mathfrak{k} \oplus \sum_{\substack{\pm \beta \in \Psi \\ \beta(Z_{\mathfrak{h}}) = 0}} \mathfrak{g}_{\beta}$$

Under our hypothesis, Proposition 3.2 says that the roots that vanish on $Z_{\mathfrak{h}}$ are compact. Whence,

$$\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} \subset \mathfrak{k} \oplus \sum_{\substack{\pm \beta \in \Psi_c \\ \beta(Z_{\mathfrak{h}}) = 0}} \mathfrak{g}_{\beta} \subset \mathfrak{k}_{\mathbb{C}}$$

Thus, the proof of the lemma is completed. \square

We now complete the proof of Proposition 3.5. We write $E_{\mathfrak{h}} = \sum_{\gamma \in \Phi_n} c_{\gamma} E_{\gamma}$ and $\text{supp}(E_{\mathfrak{h}}) = \{\gamma \in \Phi_n : c_{\gamma} \neq 0\}$. We verify that $\text{supp}(E_{\mathfrak{h}}) \subset \Psi_n$. This follows from, first $\gamma = \pm(\beta_{\ell} + \sum_{1 \leq j \leq \ell-1} n_j \beta_j)$ where $n_j \geq 0$ for all j , and second $[Z_{\mathfrak{h}}, E_{\mathfrak{h}}] = 2E_{\mathfrak{h}}$.

As before, $\mathfrak{p}^+ = \sum_{\gamma \in \Psi_n} \mathfrak{g}_{\gamma}$. Since, Ψ is holomorphic, we have \mathfrak{p}^+ is invariant under the group $\text{Ad}(K_{\mathbb{C}})$. We now show that the orbit $\text{Ad}(K_{\mathbb{C}})E_{\mathfrak{h}}$ is open in \mathfrak{p}^+ . For this, we recall the classification of the orbits of $K_{\mathbb{C}}$ in \mathfrak{p}^+ due to [RRS]. Let $S = \{\gamma_1, \dots, \gamma_r\}$ be the Harish-Chandra set of strongly orthogonal roots of Ψ_n given in (1.3). Then a set of representatives of the $K_{\mathbb{C}}$ -orbits in \mathfrak{p}^+ is

$$E_{\gamma_1} + \dots + E_{\gamma_s} \quad s = 0, 1, \dots, r$$

There is only one open orbit, which corresponds to $s = r$. Thus, there exists a $k \in K_{\mathbb{C}}$ and $1 \leq s \leq r$ so that

$$E_{\mathfrak{h}} = \text{Ad}(k)(E_{\gamma_1} + \cdots + E_{\gamma_s})$$

If it were $s < r$, then the complex simple algebra $\text{Ad}(k)(\mathfrak{sl}_2(\gamma_r))$ would be contained in the centralizer of $E_{\mathfrak{h}}$ and because of the result of Barbasch-Vogan quoted previously, we have $\text{Ad}(k)(\mathfrak{sl}_2(\gamma_r)) \subset \mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}}$. Since, γ_r is a noncompact root, and $k \in K_{\mathbb{C}}$ we would conclude that $\mathfrak{g}_{\mathbb{C}}^{\mathfrak{h}} \cap \mathfrak{p}_{\mathbb{C}}$ has positive dimension, which contradicts Lemma 3.6. Therefore $s = r$ and the orbit $\text{Ad}(K_{\mathbb{C}})E_{\mathfrak{h}}$ is open in \mathfrak{p}^+ . As a consequence we have found an element $k \in K_{\mathbb{C}}$ such that

$$\text{Ad}(k^{-1})E_{\mathfrak{h}} = E_{\gamma_1} + \cdots + E_{\gamma_r} = E_0$$

Since, the triple $\{\text{Ad}(k)Z_{\mathfrak{h}}, \text{Ad}(k)E_{\mathfrak{h}} = E_0, \text{Ad}(k)F_{\mathfrak{h}}\}$ is normal, applying Theorem 9.4.3 in [CM], there is $k' \in K_{\mathbb{C}}$ such that

$$\text{Ad}(k')Z_{\mathfrak{h}} = Z_0 \quad \text{and} \quad \text{Ad}(k')E_{\mathfrak{h}} = E_0$$

Finally we obtain the KS-triples $(Z_{\mathfrak{h}}, E_{\mathfrak{h}}, F_{\mathfrak{h}})$, (Z_0, E_0, F_0) are $K_{\mathbb{C}}$ -conjugated. The obvious map ψ_0 from the set of K_0 -conjugacy classes of KS-triples into the set of $K_{\mathbb{C}}$ -conjugacy classes of normal triples is shown to be injective in [Dk1]. Hence, there exists $k \in K_0$ which carries $Z_{\mathfrak{h}}$ onto Z_0 . Since both vectors are Ψ -dominant, it follows that they are equal. Also, k carries H onto H_0 . This ends the proof of Proposition 3.5.

The proof of the second part of Theorem 1.2 follows straightforward from Lemma 3.3 and Proposition 3.5.

The Hermitian symmetric space G/K is a *tube domain* if it is biholomorphic to a tube domain. It is well known that G/K is a tube domain if and only if the characteristic vector Z_0 defined in (1.4), is in the center of $\mathfrak{k}_{\mathbb{C}}$.

Remark 3.7. We have $\beta_{\ell}(Z_0) = 2$ and $\beta_j(Z_0) = 0$, for all $1 \leq j \leq \ell-1$, if and only if G/K is a tube domain. Whenever, G/K is not a tube domain, we have $\beta_{\ell}(Z_{\mathfrak{h}}) = 1$, and $\beta(Z_0) = 0$ for all the compact simple roots but one for which we have $\beta(Z_0) = 1$. Indeed, for a holomorphic system it happens that for any X is the center of \mathfrak{k} the value $\beta(X) = \beta_{\ell}(X)$ for any $\beta \in \Psi_n$. Also, by construction, $\beta_M \in S$ which yields $\beta_M(Z_0) = 2$. Thus, if Z_0 belongs to the center of \mathfrak{k} we have $\beta(Z_0) = 2$ for every root in Ψ_n , which gives $\beta_j(Z_0) = 0$ for every compact simple root. Certainly, the hypothesis $\beta(Z_0) = 0$ for every compact simple root, together with Ψ holomorphic yields Z_0 lies in the center of \mathfrak{k} . The hypothesis $\beta_{\ell}(Z_0) = 1$ yields Z_0 is not in the center of \mathfrak{k} which is equivalent to G/K is not a tube domain. When $\beta_{\ell}(Z_0) = 1$, since $\beta_M(Z_0) = 2$ and that the multiplicity of β_{ℓ} in β_M is one, we obtain that $\beta_j(Z_0) = 1$ for exactly one compact simple root and the root β_j has multiplicity one in the maximal root.

Remark 3.8. In [Ha], is shown a necessary condition for a nilpotent orbit to be in the wave front cycles of a tempered representation. More precisely, let L be the Levi subgroup of the centralizer of an element of a nilpotent orbit

in \mathfrak{g} and $C(G)$ the center of G , the necessary conditions is that $L/C(G)$ is be compact. Certainly the real nilpotent orbit \mathcal{O}_0 that corresponds via the Kostant-Sekiguchi map to the $K_{\mathbb{C}}$ -orbit E_0 is contained in the wave front set of any holomorphic discrete series. We have observed in a case by case computation for $BDI, EIII, EVII$ that dimension of \mathcal{O}_0 is minimal among the values of dimension of the real nilpotent orbits that satisfy the condition $L/C(G)$ is compact.

4. EXPLICIT EXAMPLES

In this section, for each Hermitian symmetric pair, we give the necessary data in order to produce an explicit example of each KS-triple $\{Z_0, E_0, F_0\}$ as well as the values $\beta(Z_0)$ for each simple root for the holomorphic system Ψ .

In [Oh] is pointed out an explicit realization of the classical real Lie algebras we are dealing with as a subalgebra of a convenient $\mathfrak{su}(a, b)$. These realization have the property that a compactly embedded Cartan subalgebra of the algebras of our interest, consists of the totality of diagonal matrices in the subalgebra. For each example on classical Lie algebras, we point out the algebra $\mathfrak{t}_{\mathbb{C}}$, a holomorphic system Ψ , the Harish-Chandra set S as in 1.3, the vector Z_0 as in (1.4), the weights $\beta_j(Z_0), j = 0, \dots, \ell$, for all β_j in (1.1), the weighted Vogan diagram that correspond to the $K_{\mathbb{C}}$ -orbit of E_0 (see [Ga]) and the signed Young diagram that corresponds to E_0 .

From the tables in [Dk1], we also present on exceptional Lie algebras the Harish-Chandra set S and the weighted Vogan diagram associated to the orbit of E_0 .

AIII, $\mathfrak{su}(p, q)$, $p < q$.

In this case

$$\mathfrak{t}_{\mathbb{C}} = \left\{ D = \text{diag}(h_1, \dots, h_p; k_1, \dots, k_q) : \sum h_j + \sum k_s = 0 \right\}.$$

We set $\epsilon_j(D) = h_j, \delta_r(D) = k_r$. Then for a holomorphic system Ψ we choose

$$\Psi_c = \{\epsilon_r - \epsilon_s, \delta_i - \delta_j, r < s, i < j\} \quad \Psi_n = \{\epsilon_i - \delta_j, 1 \leq i \leq p, 1 \leq j \leq q\}.$$

The noncompact simple root is $\beta_p = \epsilon_p - \delta_1$, other simple root we need is $\beta_q = \delta_{q-p} - \delta_{q-p+1}$.

The Harish-Chandra strongly orthogonal spanning set is

$$S = \{\epsilon_r - \delta_{q-r+1}, 1 \leq r \leq p\}.$$

The characteristic vector

$$Z_0 = \text{diag}(1, \dots, 1; 0, \dots, 0, -1, \dots, -1) \quad \text{where } \pm 1 \text{ repeats } p \text{ times.}$$

The weights $w_j = \beta_j(Z_0)$ are zero for roots other than $\beta_p, \beta_q = \delta_{q-p} - \delta_{q-p+1}$. $w_p = \beta_p(Z_0) = 1$ and $w_q = \beta_q(Z_0)$ are equal one. Whence, the weighted Vogan diagram for the orbit $K_{\mathbb{C}}E_0$ is

$$\circ - 0 \cdots 0 \quad w_p=1 \quad 0 \cdots 0 \quad w_q=1 \quad 0 \cdots 0 \quad w_{p+q-1}=0$$

The signed Young diagram for E_0 is

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Here, there are p rows of length two and $q - p$ rows of length one.

AIII, $\mathfrak{su}(p, q), p = q$.

$$\mathfrak{t}_{\mathbb{C}} = \left\{ D = \text{diag}(h_1, \dots, h_p; k_1, \dots, k_p) : \sum h_j + \sum k_s = 0 \right\}$$

We set $\epsilon_j(D) = h_j, \delta_r(D) = k_r$, with $1 \leq j, r \leq p$. Then for a holomorphic system Ψ we choose

$$\Psi_c = \{\epsilon_r - \epsilon_s, \delta_i - \delta_j, r < s, i < j\} \quad \Psi_n = \{\epsilon_i - \delta_j, 1 \leq i, j \leq p\}.$$

The noncompact simple root is $\beta_p = \epsilon_p - \delta_1$.

The Harish-Chandra strongly orthogonal spanning set is

$$S = \{\epsilon_r - \delta_{q-r+1}, 1 \leq r \leq p\}.$$

The characteristic vector is

$$Z_0 = \text{diag}(1, \dots, 1; -1, \dots, -1) \quad \text{where } \pm 1 \text{ repeats } p \text{ times}$$

Thus, the weights are $w_j = \beta_j(Z_0) = 0$ except for $w_p = \beta_p(Z_0) = 2$. So, its weighted Vogan diagram is

$$\circ - 0 \cdots 0 \quad w_p=2 \quad 0 \cdots 0 \quad 0$$

The signed Young diagram for E_0 has p rows of length two.

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BDI, $\mathfrak{so}(2p + 1, 2), p \geq 1$.

The complexification of the toroidal Cartan subalgebra is

$$\mathfrak{t}_{\mathbb{C}} = \{D = \text{diag}(h_1, \dots, h_p, -h_p, \dots, -h_1, 0, x_1, -x_1)\}$$

We set $\epsilon_j(D) = h_j, \delta_1(D) = x_1$. We fix the holomorphic system of positive roots,

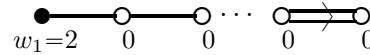
$$\Psi_c = \{\epsilon_k, \epsilon_i \pm \epsilon_j, 1 \leq k \leq p, 1 \leq i < j \leq p\} \quad \Psi_n = \{\delta_1, \delta_1 \pm \epsilon_j, 1 \leq j \leq p\}.$$

The noncompact simple root is $\beta_1 = \delta_1 - \epsilon_1$.

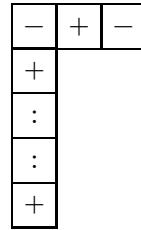
The Harish-Chandra set is $S = \{\delta_1 + \epsilon_1, \delta_1 - \epsilon_1\}$.

The characteristic vector is $Z_0 = 2H_{\delta_1} = (0, \dots, 0, 2, -2)$.

The weights of the weighted Vogan diagram are zero except the first one,



The signed Young diagram for E_0 has $2p$ rows of length one.



BDI, $\mathfrak{so}(2p, 2), p \geq 2$.

This case is similar the previous one. The Cartan subalgebra is

$$\mathfrak{t}_{\mathbb{C}} = \{D = \text{diag}(h_1, \dots, h_p, -h_p, \dots, -h_1, 0, x_1, -x_1)\}$$

We set $\epsilon_j(D) = h_j, \delta_1(D) = x_1$. The the holomorphic system we consider is

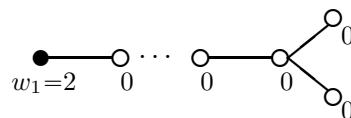
$$\Psi_c = \{\epsilon_i \pm \epsilon_j, 1 \leq i < j \leq p\} \quad \Psi_n = \{\delta_1 \pm \epsilon_j, 1 \leq j \leq p\}$$

The noncompact simple root is $\beta_p = \delta_1 - \epsilon_1$.

The Harish-Chandra set is $S = \{\delta_1 + \epsilon_1, \delta_1 - \epsilon_1\}$.

The characteristic vector is $Z_0 = 2H_{\delta_1} = (0, \dots, 0, 2, -2)$.

The weights of the weighted Vogan diagram are zero except the first one.



The signed Young diagram for E_0 has $2p - 1$ rows of length one.

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CI, $\mathfrak{sp}(n, \mathbb{R})$.

The complex Cartan subalgebra is

$$\mathfrak{t}_{\mathbb{C}} = \{D = \text{diag}(h_1, \dots, h_n, -h_n, \dots, -h_1)\}$$

We set $\epsilon_j(D) = h_j$, $1 \leq j \leq n$. The holomorphic system we consider is

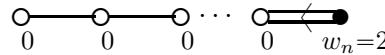
$$\Psi_c = \{\epsilon_i - \epsilon_j, 1 \leq i < j \leq n\} \quad \Psi_n = \{\epsilon_k + \epsilon_r, 1 \leq k \leq r \leq n\}$$

The noncompact simple root is $\beta_n = 2\epsilon_n$.

The set $S = \{2\epsilon_1, \dots, 2\epsilon_n\}$.

The characteristic vector is $Z_0 = (1, \dots, 1, -1, \dots, -1)$.

The weights of the weighted Vogan diagram are zero except the last one, $w_n = \beta_n(Z_0) = 2$.



The signed Young diagram for E_0 has n rows of length two.

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DIII, $\mathfrak{so}^*(2p)$, $p = 2k$.

The complex Cartan subalgebra is

$$\mathfrak{t}_{\mathbb{C}} = \{D = \text{diag}(h_1, \dots, h_p, -h_p, \dots, -h_1)\}$$

We set $\epsilon_j(D) = h_j$, $1 \leq j \leq p$. The holomorphic system we consider is

$$\Psi_c = \{\epsilon_i - \epsilon_j, 1 \leq i < j \leq p\} \quad \Psi_n = \{\epsilon_s + \epsilon_r, 1 \leq s < r \leq p\}$$

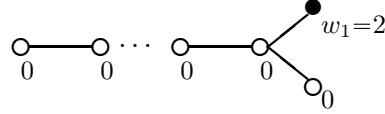
The noncompact simple root is $\beta_p = \epsilon_{p-1} + \epsilon_p$.

The Harish-Chandra set is $S = \{\epsilon_1 + \epsilon_2, \epsilon_3 + \epsilon_4, \dots, \epsilon_{2k-1} + \epsilon_{2k}\}$.

Characteristic vector is

$$Z_0 = \sum_{1 \leq j \leq k} H_{\epsilon_{2j-1} + \epsilon_{2j}} = (1, \dots, 1, -1, \dots, -1)$$

The weights $w_j = \beta_j(Z_0)$ are $w_p = 2, w_j = 0$ for $j \neq p$. So the weighted Vogan diagram is the following.



The signed Young diagram for E_0 has $2k$ rows of length two.

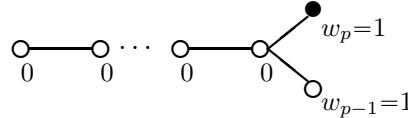
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DIII, $\mathfrak{so}^*(2p)$, $p = 2k + 1$.

This case is similar to the previous one. The difference is that the characteristic vector is

$$Z_0 = (1, \dots, 1, -1, \dots, -1, 0)$$

Thus, the weights w_{p-1}, w_p are equal to $(\epsilon_{p-1} \pm \epsilon_p)(Z_0) = 1$ and the others are zero.



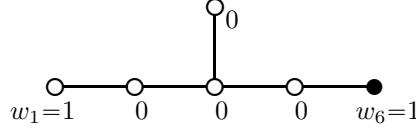
The signed Young diagram for E_0 has $2k$ rows of length two.

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EIII, $\mathfrak{e}_{6(-14)}$.

We follow the notation for the simple roots as set for Bourbaki. We fix as non compact simple root β_6 . The Harish-Chandra set in this case is $S = \{\gamma_1 = 122321, \gamma_2 = 101111\}$.

From table X of [Dk1], we extract that there is only one characteristic vector Z_h so that $\beta_\ell(Z_h) > 0$, and we obtain $Z_h = Z_0$ as in (1.4). The weighted Vogan diagram for the nilpotent orbit determinate by E_0 is

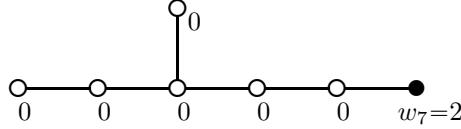


EVII, $\mathfrak{e}_{7(-25)}$.

We fix the holomorphic root system such that the noncompact simple root is α_7 . The Harish-Chandra set is

$$S = \{\gamma_1 = 22343221, \gamma_2 = 01122221, \gamma_3 = \beta_7 = 00000001\}$$

From table XIII of [Dk1], we read that the only $K_{\mathbb{C}}$ -orbit in $\mathfrak{p}_{\mathbb{C}}$ with characteristic vector $Z_{\mathfrak{h}}$ so that $\beta_{\ell}(Z_{\mathfrak{h}}) > 0$ is for $Z_{\mathfrak{h}} = Z_0$ as in 1.4. The weighted Vogan diagram is



5. MULTIPLICITIES

In this section we apply the formula for multiplicities obtained in [DV] to the particular case of the pair (G, H_0) . Henceforth, Ψ denotes a holomorphic system of positive roots for $\Phi(\mathfrak{g}, \mathfrak{t})$. To begin with we recall the necessary notation to state the results. In the notation of [DV] the pair (G, H) for our case is (G, H_0) . We have $T \subset K \subset G$ as before and $U := L := H_0 \cap K = H_0 \cap T = \exp(\mathbb{R}iZ_0)$, $\mathfrak{u} = \mathbb{R}iZ_0$, we denote by $\mathfrak{z}_{\mathfrak{k}} =$ the center of \mathfrak{k} . We define $\varphi \in \mathfrak{u}^*$ by $\varphi(Z_0) = 1$. Thus $\Phi(\mathfrak{h}_0, \mathfrak{u}) = \{\pm 2\varphi\}$. Let $\mathfrak{k}_{\mathfrak{z}} = \mathfrak{k}^{Z_0}$ denote the centralizer of Z_0 in \mathfrak{k} and $\Phi_{\mathfrak{z}}$ the root system for $(\mathfrak{k}_{\mathfrak{z}}, \mathfrak{t})$. Thus,

$$\Phi_{\mathfrak{z}} = \{\alpha \in \Phi(\mathfrak{k}, \mathfrak{t}) : \alpha(\mathfrak{u}) = 0\}$$

By Remark 3.7, if the Hermitian symmetric space G/K is a tube domain, then $Z_0 \in \mathfrak{z}_{\mathfrak{k}}$, $\Phi_{\mathfrak{z}} = \Phi(\mathfrak{k}, \mathfrak{t})$ and the analytic subgroup of G with Lie algebra $\mathfrak{k}_{\mathfrak{z}}$ is $K_{\mathfrak{z}} = K$. If G/K is not a tube domain, $Z_0 \notin \mathfrak{z}_{\mathfrak{k}}$, then owing to $\beta_j(Z_0) = 0$ for all compact simple roots but one, the semisimple factor of $\mathfrak{k}_{\mathfrak{z}}$ has rank $\ell - 2$. The list of the triples $(\mathfrak{g}, \mathfrak{k}, \mathfrak{k}_{\mathfrak{z}})$ that not correspond to tube domains is:

\mathfrak{g}	$\mathfrak{su}(p, q), p < q$	$\mathfrak{so}^*(2(2k+1))$	$\mathfrak{e}_{6(-14)}$
\mathfrak{k}	$\mathfrak{su}(p) + \mathfrak{su}(q) + \mathfrak{z}_{\mathfrak{k}}$	$\mathfrak{su}(2k+1) + \mathfrak{z}_{\mathfrak{k}}$	$\mathfrak{so}(10) + \mathfrak{z}_{\mathfrak{k}}$
$\mathfrak{k}_{\mathfrak{z}}$	$\mathfrak{su}(p) + \mathfrak{su}(q-p) + \mathfrak{su}(p) + \mathfrak{t}$	$\mathfrak{su}(2k) + \mathfrak{t}$	$\mathfrak{so}(8) + \mathfrak{t}$

The set of equivalence classes of irreducible square integrable representations of G is parameterized by the set of Harish-Chandra parameters. These

parameters are $\lambda \in it^*$ so that $\lambda + \rho$ lifts to a character of T , here ρ is equal to one half of the sum of the elements in Ψ ,

$$(5.1) \quad (\lambda, \alpha) > 0 \text{ for all } \alpha \in \Psi_c \quad \text{and} \quad (\lambda, \alpha) \neq 0 \text{ for all } \alpha \in \Phi_n.$$

The holomorphic irreducible square integrable representation corresponds to parameters which also satisfy $(\lambda, \alpha) > 0$ for all $\alpha \in \Psi_n$.

The set of Harish-Chandra parameters which corresponds to the irreducible square integrable representations of H_0 is $P := \{n\varphi : n \in \mathbb{Z} \setminus \{0\}\}$.

The set of Harish-Chandra parameters for a compact Lie group L is equal to the set of strictly dominant integral weights for L , equivalently, the set of Harish-Chandra parameters is equal to the set of infinitesimal character of the set of irreducible representations of L . We denote the parametrization by $\mu \leftrightarrow (\pi_\mu^L, V_\mu^L)$.

Let $(\pi_\lambda, V_\lambda^G)$ be a holomorphic irreducible square integrable representation. Therefore, the restriction $\text{res}_{H_0}(\pi_\lambda)$ of π_λ to the subgroup H_0 is an admissible discretely decomposable representation of H_0 . For $\mu \in P$, let $(\sigma_\mu, V_\mu^{H_0})$ denote the irreducible square integrable representation of H_0 of Harish-Chandra parameter μ . Let $m(\pi_\lambda, \sigma_\mu) = \text{Hom}_{H_0}(\sigma_\mu, \pi_\lambda)$ denote the multiplicity of σ_μ in $\text{res}_{H_0}(\pi_\lambda)$. Therefore, we have a Hilbert discrete sum decomposition

$$\text{res}_{H_0}(\pi_\lambda) = \sum_{\mu \in P} m(\pi_\lambda, \sigma_\mu) V_\mu^{H_0}$$

We write the restriction to $K_\mathfrak{z}$ of the lowest K -type $\pi_{\lambda+\rho_n}^K$ for $(\pi_\lambda, V_\lambda^G)$ as

$$\text{res}_{K_\mathfrak{z}}(\pi_{\lambda+\rho_n}^K) = \sum_{1 \leq j \leq s} m(\pi_{\lambda+\rho_n}^K, \pi_{\mu_j+\rho_\mathfrak{z}}^{K_\mathfrak{z}}) V_{\mu_j+\rho_\mathfrak{z}}^{K_\mathfrak{z}}$$

where ρ_n is the half sum of the roots in Ψ_n and $\rho_\mathfrak{z}$ is the half sum of the roots in $\Phi_\mathfrak{z} \cap \Psi$.

Theorem 5.1.

(i) $m(\pi_\lambda, \sigma_\mu) > 0$ if and only if μ belongs to the set

$$\{[(\mu_j + \rho_\mathfrak{z})(Z_0) + n - 1]\varphi : 1 \leq j \leq s, n \geq 0\}$$

(ii) The multiplicity $m(\pi_\lambda, \sigma_{m\varphi})$ is equal to

$$\sum_{\substack{j, n \\ \mu_j(Z_0) + n = m}} m(\pi_{\lambda+\rho_n}^K, \pi_{\mu_j+\rho_\mathfrak{z}}^{K_\mathfrak{z}}) \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2h+c-1}{c-1} \binom{h+d-1}{d-1}.$$

Here $c = |\{\gamma \in \Psi_n : \gamma(Z_0) = 1\}|$ and $d+1 = |\{\gamma \in \Psi_n : \gamma(Z_0) = 2\}|$.

Remark 5.2. By work of M. Vergne, Jacobsen, Oshima and Möller, it follows that $\text{res}_{H_0}(\pi_\lambda)$ is equal to a discrete Hilbert sum of holomorphic representations, our contribution is to determinate the Harish-Chandra parameters for H_0 that contributes and the respective multiplicities. However, we obtain an independent proof of the result.

The proof of the theorem will take up the rest of this section. It requires more notation. To start, we consider the restriction $q_{\mathfrak{u}} : \mathfrak{t}^* \rightarrow \mathfrak{u}^*$ and the multiset $\Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{u}) := q_{\mathfrak{u}}(\Psi(\mathfrak{k}, \mathfrak{t}) \setminus \Phi_{\mathfrak{z}})$. So, we have

$$(5.2) \quad \Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{u}) = \begin{cases} \emptyset & \text{if } G/K \text{ is a tube domain} \\ \underbrace{\{\varphi, \dots, \varphi\}}_a & \text{if } G/K \text{ is not a tube domain} \end{cases}$$

Here $a = |\{\alpha \in \Psi_c : q_{\mathfrak{u}}(\alpha)(Z_0) = 1\}| = \frac{1}{2} \dim(K/K_{\mathfrak{z}})$. In fact, when G/K is a tube domain then $iZ_0 \in \mathfrak{z}_{\mathfrak{k}}$ and the first claim is obvious. For a not tube domain Remark 3.7 yields that $\alpha(Z_0) = 1$ or $\alpha(Z_0) = 0$ for $\alpha \in \Psi_c$. For w in the Weyl group W_K of K , we compute the multiset

$$S_w^{H_0} := [q_{\mathfrak{u}}(w\Psi_n) \cup \Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{u})] \setminus \Phi(\mathfrak{h}_0, \mathfrak{u}).$$

Since, Ψ is holomorphic and $w \in W_K$ it follows that $w\Psi_n = \Psi_n$. Hence, $S_w^{H_0}$ does not depend on w . We have,

$$(5.3) \quad q_{\mathfrak{u}}(\Psi_n) = \underbrace{\{\varphi, \dots, \varphi\}}_c \cup \underbrace{\{2\varphi, \dots, 2\varphi\}}_{d+1}.$$

If G/K is a tube domain, then $c = 0$ and $d+1 = |\Psi_n|$. Indeed, in Remark 3.7 we show $\gamma(Z_0) = 2$ for all noncompact positive root. Therefore, from (5.2), (5.3) and the previous computation, for a tube domain we obtain,

$$(5.4) \quad S_w^{H_0} = \underbrace{\{2\varphi, \dots, 2\varphi\}}_d.$$

If G/K is not a tube domain the values of c and d are in the following table.

\mathfrak{g}	$\mathfrak{su}(p, q)$, $p < q$	$\mathfrak{so}^*(2(2k+1))$	$\mathfrak{e}_{6(-14)}$
c	$(q-p)p$	$2k$	8
$d+1$	p^2	$k(2k-1)$	8

Hence,

$$(5.5) \quad S_w^{H_0} = \underbrace{\{\varphi, \dots, \varphi\}}_c \cup \underbrace{\{2\varphi, \dots, 2\varphi\}}_d \cup \Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{u}).$$

For $\nu \in i\mathfrak{t}^*$ (resp. $\nu \in i\mathfrak{u}^*$), δ_{ν} denotes the Dirac distribution on $i\mathfrak{t}^*$ (resp. on $i\mathfrak{u}^*$) defined by ν . Let $(q_{\mathfrak{u}})_*(\delta_{\nu})$ be the push-forward of δ_{ν} from $i\mathfrak{t}^*$ to $i\mathfrak{u}^*$. Thus, $(q_{\mathfrak{u}})_*(\delta_{\nu}) = \delta_{\nu}$. Let

$$y_{\nu} = \sum_{n=0}^{\infty} \delta_{n\nu + \frac{\nu}{2}}, \quad z_{\nu} = \sum_{n=0}^{\infty} \delta_{n\nu}.$$

For a strict finite set $T = \{\nu_1, \dots, \nu_t\} \subset i\mathfrak{t}^*$ we define

$$y_T = y_{\nu_1} * \dots * y_{\nu_t} = \underset{\nu \in T}{\ast} y_{\nu}.$$

Here $*$ means convolution of distributions. Let $\varpi_{\mathfrak{z}}(\lambda) := \prod_{\alpha \in \Psi \cap \Phi_{\mathfrak{z}}} \frac{\lambda(\alpha)}{\rho_{\mathfrak{z}}(\alpha)}$. Then, in [DV] is shown the following equality of distributions on $i\mathfrak{u}^*$,

$$(5.6) \quad \sum_{\mu \in P} m(\pi_{\lambda}, \sigma_{\mu}) \delta_{\mu} = \sum_{w \in W_{\mathfrak{z}} \setminus W_K} \epsilon(w) \varpi_{\mathfrak{z}}(w\lambda) \delta_{q_{\mathfrak{u}}(w\lambda)} * y_{S_w^{H_0}}.$$

where $\epsilon(w)$ is the sign of w . The validity of the above equality follows by Lemma 3.3 in [DV] because Condition (C) is satisfied. We now show Theorem 5.1 for G so that G/K is a tube domain. For a holomorphic system Ψ we always have the equality

$$(5.7) \quad \varpi_{\mathfrak{z}}(w\lambda) = \varpi_{\mathfrak{z}}(w(\lambda + \rho_n)).$$

Now $\mathfrak{k}_{\mathfrak{z}} = \mathfrak{k}$, hence we have $\varpi_{\mathfrak{z}}(\lambda) = \dim V_{\lambda + \rho_n}^K$ which is equal to the dimension of lowest K -type of π_{λ} . Then, by (5.6), (5.4) together and the above consideration gives

$$\sum_{\mu \in P} m(\pi_{\lambda}, \sigma_{\mu}) \delta_{\mu} = \dim V_{\lambda + \rho_n}^K \delta_{\lambda(Z_0)\varphi} * y_{2\varphi}^{|\Psi_n|-1}.$$

Obviously, $y_{\nu} = \delta_{\frac{\nu}{2}} * z_{\nu}$, $q_{\mathfrak{u}}(\rho_n) = |\Psi_n| \varphi$ and

$$y_{2\varphi}^{|\Psi_n|-1} = z_{2\varphi}^{|\Psi_n|-1} * \delta_{-\varphi} * \delta_{q_{\mathfrak{u}}(\rho_n)}.$$

For r, s positive integers, it readily follows that

$$(5.8) \quad z_{r\varphi}^s = \sum_{t=0}^{\infty} \binom{t+s-1}{s-1} \delta_{tr\varphi}.$$

Hence, we obtain

$$\sum_{\mu \in P} m(\pi_{\lambda}, \sigma_{\mu}) \delta_{\mu} = \sum_{t \geq 0} \dim V_{\lambda + \rho_n}^K \binom{t + |\Psi_n| - 2}{|\Psi_n| - 2} \delta_{[(\lambda + \rho_n)(Z_0) + 2t - 1]\varphi}.$$

Therefore, whenever G/K is a tube domain, the Harish-Chandra parameters that contribute to $\text{res}_{H_0}(\pi_{\lambda})$ are $[(\lambda + \rho_n)(Z_0) + 2t - 1]\varphi = [\lambda(Z_0) + d + 2t]\varphi$, $t = 0, 1, \dots$, and the respective multiplicities are exactly the numbers $\binom{t + |\Psi_n| - 2}{|\Psi_n| - 2} \dim V_{\lambda + \rho_n}^K = \binom{t+d-1}{d-1} \dim V_{\lambda + \rho_n}^K$.

To follow, we show the multiplicity formula when G/K is not a tube domain. Hence, $K_{\mathfrak{z}}$ is a proper subgroup of K and iZ_0 is not in the center of \mathfrak{k} . We manipulate on the right hand side of formulae (5.6). Since Ψ is a holomorphic system, $w\rho_n = \rho_n$ for $w \in W_K$. Hence, $q_{\mathfrak{u}}(\rho_n) = [\frac{c}{2} + d + 1]\varphi$ and $y_{q_{\mathfrak{u}}(\Psi_n) \setminus \Phi(\mathfrak{h}_0, \mathfrak{u})} = z_{q_{\mathfrak{u}}(\Psi_n) \setminus \Phi(\mathfrak{h}_0, \mathfrak{u})} * \delta_{[\frac{c}{2} + d]\varphi}$. Hence, the right hand side of (5.6) becomes equal to

$$(5.9) \quad \sum_{w \in W_{\mathfrak{z}} \setminus W_K} \epsilon(w) \varpi_{\mathfrak{z}}(w(\lambda + \rho_n)) \delta_{q_{\mathfrak{u}}(w(\lambda + \rho_n))} * \underset{\gamma \in q_{\mathfrak{u}}(\Psi \setminus \Phi_{\mathfrak{z}})}{\ast} y_{\gamma} * \underset{\beta \in q_{\mathfrak{u}}(\Psi_n) \setminus \Phi(\mathfrak{h}_0)}{\ast} z_{\beta} * \delta_{-\varphi}.$$

In the language of discrete Heaviside distributions, the restriction of the lowest K -type $\pi_{\lambda+\rho_n}^K$ of π_λ to U is represented by

$$(5.10) \quad \sum_{w \in W_{\mathfrak{J}} \setminus W_K} \epsilon(w) \varpi_{\mathfrak{J}}(w(\lambda + \rho_n)) \delta_{q_{\mathfrak{u}}(w(\lambda + \rho_n))} * \underset{\gamma \in q_{\mathfrak{u}}(\Psi_c \setminus \Phi_{\mathfrak{J}})}{*} y_\gamma$$

The restriction of $\pi_{\lambda+\rho_n}^K$ to U can be represented as the restriction of $\pi_{\lambda+\rho_n}^K$ to $K_{\mathfrak{J}}$ and then we decompose the resulting representation of $K_{\mathfrak{J}}$ as U -module. Let $\mu_1 + \rho_{\mathfrak{J}}, \dots, \mu_s + \rho_{\mathfrak{J}}$, denote the infinitesimal characters for the irreducible constituents of $\text{res}_{K_{\mathfrak{J}}}(\pi_{\lambda+\rho_n}^K)$. Here we take μ_j dominant with respect to $\Psi \cap \Phi_{\mathfrak{J}}$. Then, we have the equality

$$\text{res}_U(\pi_{\lambda+\rho_n}^K) = \sum_{1 \leq j \leq s} m(\pi_{\lambda+\rho_n}^K, \pi_{\mu_j + \rho_{\mathfrak{J}}}^{K_{\mathfrak{J}}}) V_{\mu_j + \rho_{\mathfrak{J}}}^U.$$

Therefore, we have that (5.10) is equal to

$$(5.11) \quad \sum_{1 \leq j \leq s} m(\pi_{\lambda+\rho_n}^K, \pi_{\mu_j + \rho_{\mathfrak{J}}}^{K_{\mathfrak{J}}}) \delta_{(\mu_j + \rho_{\mathfrak{J}})(Z_0)\varphi} *$$

Putting together the new expression for (5.10) and (5.9), we obtain that the right hand side of (5.6) is equal to

$$\sum_{1 \leq j \leq s} m(\pi_{\lambda+\rho_n}^K, \pi_{\mu_j + \rho_{\mathfrak{J}}}^{K_{\mathfrak{J}}}) \delta_{(\mu_j + \rho_{\mathfrak{J}})(Z_0)\varphi} * \underset{\gamma \in q_{\mathfrak{u}}(\Psi_n) \setminus \Phi(\mathfrak{h}_0, \mathfrak{u})}{*} z_\gamma * \delta_\varphi.$$

After we recall (5.5) and we apply (5.11) to the previous formula, we obtain

$$(5.12) \quad \sum_{\mu \in P} m(\pi_\lambda, \sigma_\mu) \delta_\mu = \sum_{t \geq 0, h \geq 0} \left[\sum_j m(\pi_{\lambda+\rho_n}^K, \pi_{\mu_j + \rho_{\mathfrak{J}}}^{K_{\mathfrak{J}}}) \binom{t+c-1}{c-1} \binom{h+d-1}{d-1} \delta_{[(\mu_j + \rho_{\mathfrak{J}})(Z_0) + t + 2h - 1]\varphi} \right]$$

Hence,

$$(5.13) \quad \sum_{\mu \in P} m(\pi_\lambda, \sigma_\mu) \delta_\mu = \sum_{j=1}^s m(\pi_{\lambda+\rho_n}^K, \pi_{\mu_j + \rho_{\mathfrak{J}}}^{K_{\mathfrak{J}}}) \sum_{n \geq 0} \sum_{h=0}^{\left[\frac{n}{2}\right]} \binom{n-2h+c-1}{c-1} \binom{h+d-1}{d-1} \delta_{[(\mu_j + \rho_{\mathfrak{J}})(Z_0) + n - 1]\varphi}.$$

Therefore a Harish-Chandra parameter $m\varphi$ of an irreducible H_0 -factors for $\text{res}_{H_0}(\pi_\lambda)$ belongs to the set

$$\{[(\mu_j + \rho_{\mathfrak{J}})(Z_0) + n - 1]\varphi : n = 0, 1, \dots, j = 1, \dots, s\},$$

and the respective multiplicity is

$$\sum_{\substack{j,n \\ \mu_j(z_0)+n=m}} m(\pi_{\lambda+\rho_n}^K, \pi_{\mu_j+\rho_j}^{K_3}) \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2h+c-1}{c-1} \binom{h+d-1}{d-1}.$$

Now, the proof of Theorem 5.1 has been completed.

Remark 5.3. In the above formulae for either Harish-Chandra parameters or multiplicities, if we make c equal to zero, we obtain the formula for the tube type case. This is the reason we have left the summand $\rho_3(Z_0)$ even though it is equal to zero when G/K is a tube type domain.

Remark 5.4. The decomposition of the adjoint representation of $\mathfrak{g}_{\mathbb{C}}$ restricted to \mathfrak{h}_0 is,

(i) When G/K is a tube domain,

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_1^{d+1} (\mathbb{C}^3, 2\varphi) \oplus \bigoplus_1^{\dim \mathfrak{k} - d - 1} (\mathbb{C}, 0\varphi)$$

(ii) When G/K is not a tube domain,

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_1^{d+1} (\mathbb{C}^3, 2\varphi) \oplus \bigoplus_1^c (\mathbb{C}^2, \varphi) \oplus \bigoplus_1^{\dim \mathfrak{k} - d - 1} (\mathbb{C}, 0\varphi)$$

Whence, the coefficients c and $d+1$ represent the multiplicity of the distinct irreducible constituents of the \mathfrak{h}_0 -module $\mathfrak{g}_{\mathbb{C}}$.

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FAMAF-CIEM, CIUDAD UNIVERSITARIA, 5000 CÓRDOBA, ARGENTINE
E-mail address: galina@famaf.unc.edu.ar, vargas@famaf.unc.edu.ar